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Orthogonal Polynomials and Positivity

by

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1. Introduction.

The literature of special functions is full of explicit formulas, some extremely useful and some of no value at all. While much of the rest of mathematics is no longer as constructive as it was in the eighteenth century, this is not true in the theory of special functions. Even the extremely interesting recent work on Lie theory has been primarily concerned with specific formulas and has concentrated on general methods to obtain these formulas. I have no quarrel with this general tendency, since it is usually possible to prove more if a specific formula is obtained than if nothing more than an existence theorem is proven. However, there are times when the specific formulas obtained are not useful for certain problems, while a general property, say the positivity of some kernel, is the essential fact that is needed. We will give some instances of this, at times when we can not find a specific formula at all and other examples when a specific formula can be found but the positivity must be obtained by a different argument.

2. Linearization.

We start with a problem that has examples of both of these features. Recall that

$$\cos n\theta \qquad \cos m\theta = \frac{1}{2}\left[\cos(n+m)\theta + \cos(n-m)\theta\right].$$

Since $T_n(\cos\theta) = \cos n\theta$ we have

$$T_n(x)T_m(x) = \frac{1}{2}\left[T_{n+m}(x) + T_{|n-m|}(x)\right] . \tag{2.1}$$

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$T_n(x)$ are orthogonal polynomials and we can ask if there is a generalization of(2.1) to a wide class of orthogonal polynomials. In one sense we can generalize (2.1) to all orthogonal polynomials $\{p_m(x)\}$ for

$$(2.2) \qquad p_n(x)p_m(x) = \sum_{k=n-m}^{m+m} \alpha_k p_k(x).$$

However, for certain problem it is essential that $\alpha_k \geq 0$ as it is in (2.1). One example is the following . Let $\{p_n(x)\}$ be orthogonal on $[-1,1]$ with respect to a measure $d\alpha(x)$, assume that they are normalized by $p_n(1)=1$, and that $\alpha_k \geqq 0$ in (2.2). Then if $P(x) = \sup\limits_{n=0,1,..} |p_n(x)|$ and $M = \sup\limits_{-1\leqq x\leqq 1} P(x)$ we have $P^2(x) \leqq M$, since $\sum \alpha_k = \sum |\alpha_k| = 1$.

Thus $M^2 \leqq M$ so $M = 1$, or

$$(2.3) \qquad |p_n(x)| \leqq p_n(1), \quad -1 \leqq x \leqq 1.$$

(2.3) is a property of many of the classical polynomials, for example the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha \geqq \beta$, $\alpha \geqq -\frac{1}{2}$, where now $d\alpha(x) = (1-x)^\alpha(1+x)^\beta$,$\alpha,\beta > -1$. See [40,(7,32,2)] .

The proof Szegö gives uses the differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$. This proof will not extend to other orthogonal polynomials since it is well known that orthogonal polynomials in general do not satisfy a differential equation that is simple enough to deal with. In fact, essentially the only orthogonal polynomials that satisfy a reasonable differential equation are the classical polynomials, Jacobi polynomials and their limiting cases, Laguerre and Hermite polynomials, [17] . However, there are a number of other special orthogonal polynomials which are of interest and the inequality (2.3) is a useful inequality to have. If we can obtain (2.2) for a fairly general class of polynomials we will have some new instances of (2.3).

There are a number of other applications of (2.2) with $\alpha_k \geqq 0$. After we state and prove a theorem which implies $\alpha_k \geqq 0$ in (2.2) we will mention some of these applications.

For the time being we will normalize our polynomials by

$p_n(x) = x^n + \ldots$ Then if $m = 1$, (2.2) becomes

$$p_1(x)p_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x). \tag{2.4}$$

Favard [18] and Shohat [39] have shown that a necessary and sufficient condition that $\{p_n(x)\}$ are a set of orthogonal polynomials is that $p_1(x) = x + c$, c real, α_n real, and $\beta_n > 0$. If (2.2) is to hold with non-negative coefficients we must have $\alpha_n \geqq 0$. The following theorem gives a sufficient condition.

<u>Theorem 1.</u> Let $p_0(x) = 1$, $p_1(x) = x + c$, c real, $p_1(x)p_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x)$. Then if $\alpha_n \geqq 0$, $\beta_{n+1} > 0$, $\alpha_{n+1} \geqq \alpha_n$, $\beta_{n+2} \geqq \beta_{n+1}$, $n = 0,1,\ldots$, we have

$$p_n(x)p_m(x) = \sum_{k=n-m}^{n+m} \alpha_{k,m,n}\, p_k(x) \tag{2.5}$$

with $\alpha_{k,m,n} \geqq 0$.

There is a more general theorem for difference equations which implies Theorem 1.

<u>Theorem 2</u>. Define Δ_n by $\Delta_n k(n) = k(n+1) + \alpha_n k(n) + \beta_n k(n-1)$. Let $a(n,m)$ satisfy the equation

$$\Delta_n a(n,m) = \Delta_m a(n,m),$$

with

$$a(n,0) = a(0,n) = a(n),$$
$$a(n,-1) = a(-1,n) \leqq 0.$$

If $a(n) \geqq 0$ and $\alpha_{n+1} \geqq \alpha_n$, $\beta_n \geqq 0$, $\beta_{n+1} \geqq \beta_n$, $n = 0,1,$

then

$$a(n,m) \geqq 0, \quad n,m = 0,1,\ldots,$$

Theorem 1 follows from Theorem 2 when $a(-1,n) = 0$, $\beta_{n+1} > 0$ and $\alpha_n \geqq 0$, $n = 0,1,\ldots$, The proof of Theorem 2 is identical with the proof of Theorem 1 that was given in [4] , but it is so simple and the method is useful in other problems so we repeat it here.

By symmetry it is sufficient to prove $a(m,n) \geqq 0$ for $m \leqq n$. We assume that $a(k,n) \geqq 0$ for $k = 0,1,\ldots,m$ and consider $a(m+1,n)$. By (2.4) we have $a(m+1,n)+\alpha_m a(m,n)+\beta_m a(m-1,n)=a(m,n+1)+\alpha_n a(m,n)+\beta_n a(m,n-1)$, so

$$(2.6)\quad a(m+1,n)=a(m,n+1)+(\alpha_n-\alpha_m)a(m,n)+(\beta_n-\beta_m)a(m,n-1)+\beta_m\left[a(m,n-1)-a(m-1,n)\right].$$

If we can show that $a(m,n-1)-a(m-1,n) \geqq 0$ we are done, since all the terms on the right are then non-negative. But (2.6) is equivalent to $a(m+1,n)-a(m,n+1)=(\alpha_n-\alpha_m)a(m,n)+(\beta_n-\beta_m)a(m,n-1)+\beta_m\left[a(m,n-1)-a(m-1,n)\right]$, and the assumptions $a(0,k) \geqq 0$, $a(-1,k) \leqq 0$ show that

$$(2.7)\quad a(m+1,n) \geqq a(m,n+1) \geqq \ldots \geqq a(0,n+m+1) \geqq 0.$$

This completes the proof of Theorem 2 and also shows why this theorem is not able to give us all the results that are known for the classical polynomials. Not only have we shown that $a(m,n) \geqq 0$ but we have shown that $a(m,n)$ has a type of monotonicity given by (2.7). This property is not not always satisfied for the classical polynomials. It would be of real interest to find a theorem which would give us the conclusion of Theorem 1 for all the classical polynomials or even the conclusion in the easier case $\alpha_n=0$. One possible method is to normalize the polynomials differently so that a different equation is satisfied by the coefficients. Then these differently normalized coefficients may be monotone and a theorem of the sort we just proved may be possible. One other possible theorem is a comparison type theorem, since we know the theorem when $\beta_1 = \frac{1}{2}$; $\beta_n = \frac{1}{4}$, $n = 2,3,\ldots$ This is the case $p_n(x) = 2^{1-n}T_n(x)$. There may be a maximum theorem for equations that are in some sense between this equation and those of Theorem 2.

A number of examples of Theorem 1 for specific polynomial sets are given given in [4]. These include Jacobi, Laguerre, Hermite, Charlier, and Meixner polynomials. In all of these cases except Jacobi polynomials we obtain non-negativity of the coefficients for all possible values of the parameter. In the case of Jacobi polynomials Theorem 1 applies to a wide class but it does not tell the whole story. Gasper, [22], [23], has proven the following result.

<u>Theorem 3</u>. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial, i.e. the orthogonal polynomials with respect to $(1-x)^{\alpha}(1+x)^{\beta}$, normalized by $P_n^{(\alpha,\beta)}(1) > 0$. Then

$$P_n^{(\alpha,\beta)}(x)\ P_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} \alpha_{k,m,n}\ P_k^{(\alpha,\beta)}(x)$$

with $\alpha_{k,m,n} \geqq 0$ if $\alpha \geqq \beta > -1$ and

$$(\alpha+\beta+1)(\alpha+\beta+6)(\alpha+\beta+4)^2 \geqq [(\alpha+\beta+1)^2 - 7\ (\alpha+\beta+1) - 4]\ (\alpha-\beta)^2,$$

and some $\alpha_{k,m,n} < 0$ if this condition fails to hold. For $\alpha < \beta$ there is a similar result if the polynomials are normalized by $P_n^{(\alpha,\beta)}(-1) > 0$.

The region in Theorem 3 includes $\alpha \geqq \beta, \alpha + \beta + 1 \geqq 0$, while in Theorem 1 it only implies this result for a region slightly larger rhan $\alpha + \beta - 1 \geqq 0$. In particular for $\alpha = \beta$ Theorem 1 does not imply the classical result for $-\frac{1}{2} \leqq \alpha < \frac{1}{2}$. A necessary condition for Theorem 1 to hold if $\alpha_k = 0$ is easily seen to be

$$\beta_n + \beta_{n+1} + \dots + \beta_{n+m} - \beta_1 - \dots - \beta_m \geqq 0. \tag{2.8}$$

This is obtained by computing the coefficient of $p_{n+m-2}(x)$ in (2.5). However (2.8) is far from sufficient. If

$$\beta_1 = \frac{1}{2} - \epsilon;\ \beta_2 = \frac{1}{4} + \epsilon;\beta_n = \frac{1}{4},\ n = 3,4,\dots,$$

then the coefficient of $p_n(x)$ in $p_4(x)p_n(x)$ is negative for $0 < \epsilon < \frac{1}{2}$. The interest in this example is that it has a large number of positive successive differences. For $P_n^{(\alpha,\alpha)}$, $-\frac{1}{2} \leqq \alpha \leqq \frac{1}{2}$, β_n has positive successive differences of all orders and so is a moment sequence on $[0,1]$,

$$\beta_n = \int_0^1 t^n d\mu(t),\ d\mu \geqq 0,\ n = 1,2,\dots\ . \tag{2.9}$$

I know of no examples of β_n given by (2.9) for which the necessary condition (2.8) holds and yet the conclusion of Theorem 1 fails. This suggests that polynomials with $\alpha_n = 0$, β_n given by (2.9) are worth some serious study. They remind us of Fejér's generalized Legendre polynomials. A summary of his work on them and other results of Szegö is given in Szegö [40].

We give one more polynomial set that satisfies the assumptions of Theorem 1. In a series of short notes Pollaczek has defined a number of very interesting classes of orthogonal polynomials which generalize the classical polynomials and have very singular behavior, [17], [40]. The easiest generalization of the ultraspherical polynomials of this type is given by

$$x\, R_n^\lambda(x;a) = R_{n+1}^\lambda(x;a) + \frac{(n+2\lambda-1)n}{4(n+\lambda+a)(n+\lambda+a-1)} R_{n-1}^\lambda(x;a) ,$$

which reduces to the ultraspherical polynomials when $a = 0$. We have $\beta_n > 0$, $n = 1,2, \ldots$, for $\lambda > 0$, $\lambda + a > 0$; $a = 0$, $\lambda > -\frac{1}{2}$; or $-\frac{1}{2} < \lambda < 0, -1 < a + \lambda < 0$. An easy computation shows that $\beta_{n+1} \geqq \beta_n$ if $a \geqq 0$, $a \geqq (\lambda-\lambda^2)/(1+\lambda)$. Thus we have

$$(2.10) \qquad |R_n^\lambda(x;a)| \leqq R_n^\lambda(1;a), \; -1 \leqq x \leqq 1, \; a \geqq 0, \; a \geqq (\lambda-\lambda^2)/(1+\lambda), \quad \lambda > a.$$

(2.10) also holds for $R_n^\lambda(x;a)$ for $0 < \lambda < 1$, $a \geqq 0$ as will be shown in [5] using a theorem that we will state in the next section.

We mentioned before stating Theorem 1 that there is sometimes a specific formula connected with this problem. For the classical polynomials this is true. In particular, for Jacobi polynomials the coefficients are Appell generalized hypergeometric functions of two variables evaluated at $x = y = 1$, [33]. Unfortunately it seems to be impossible to use this formula to prove the positivity. However there are a number of cases where we have not obtained explicit formulas, for instance for Charlier, Meixner, and Pollaczek polynomials. This should be possible at least in the Charlier case.

The most interesting application of Theorem 1 is to the construction of new Banach algebras and then the applications that can be made of this Banach algebra structure. We get a Banach algebra structure if our polynomials are orthogonal on a compact set, which it is if and only if $|\alpha_n| \leqq A$, $0 < \beta_n \leqq B$. If we also assume monotonicity as we did in Theorem 1 then this set is a finite interval $[a,b]$. See [15]. We have $p_n(b)>0$ since all the zeros of $p_n(x)$ lie in (a,b) and $p_n(x) = x^n + \ldots$ is positive for x large. Let $r_n(x)$ be these polynomials renormalized by $r_n(b) = 1$. Define the Fourier coefficient c_n by

$$c_n = \int_a^b f(x)\ r_n(x) d\alpha(x) ,$$

where $d\alpha(x)$ is the measure for which $r_n(x)$ are orthogonal. Then

$$f(x) \sim \sum c_n h_n r_n(x)$$

where

$$h_n^{-1} = \int_a^b [r_n(x)]^2\ d\alpha(x)$$

Define the generalized translate of c_n by

$$c_{n,m} = \int_a^b f(x)\ r_n(x)\ r_m(x)\ d\alpha(x)$$

Then $c_{n,0} = c_n$ and $c_{n,m} \geqq 0$ if $c_n \geqq 0$. Define the l^1 norm by

$$||c_n||_1 = \sum_{n=0}^{\infty} |c_n| h_n$$

and the convolution by

$$d_n = \sum_{m=0}^{\infty} c_{n,m}\, b_m\, h_m.$$

Then

$$||d_n||_1 \leqq \sum_{n=0}^{\infty} h_n \sum_{m=0}^{\infty} |b_m| h_m \sum_{k=0}^{\infty} \alpha_{k,m,n} |c_k| h_k$$

where $\alpha_{k,m,n} = 0$ if $k > m + n$, $m > n + k$, or $n > m + k$. Then

$$||d_n||_1 \leqq \sum_{k=0}^{\infty} |c_k| h_k \sum_{m=0}^{\infty} |b_m| h_m \sum_{n=0}^{\infty} \alpha_{k,m,n}\, h_n$$

since $\alpha_{k,m,n} \geqq 0$.

Also

$$r_k(x)\ r_m(x) = \sum_{n=0}^{\infty} \alpha_{k,m,n}\ h_n\ r_n(x)$$

so

$$\sum_{n=0}^{\infty} \alpha_{k,m,n}\ h_n = 1$$

Thus

$$||d_n||_1 \leqq ||c_n||_1 \quad ||b_n||_1$$

Similarly we can show

$$||d_n||_\infty \leqq ||c_n||_\infty \quad ||b_n||_1$$

where

$$||d_n||_\infty = \sup|d_n|.$$

Then we not only have a Banach algebra, but a convolution algebra in the sense of O'Neil [35] when we define

$$||d_n||_p = \left[\sum_{n=0}^{\infty} |d_n|^p\ h_n\right]^{\frac{1}{p}}.$$

See [16] and [8] for an application to Toeplitz determinants. A stochastic application of this result for ultraspherical polynomials is given by Kennedy [29]. This can be extended to Jacobi polynomials using Gasper's result mentioned above.

One last remark is to repeat a remark made in [4]. The assumptions in Theorem 1 are implied by the following condition on birth and death processes. The probability of a transition to the right at the stage n + 1 is at least as great as the probability of a transition to the right at stage n and the same for movement to the left. This is a very natural assumption to make on a birth and death process, i.e. the larger the population the more chance of a single birth or death, and it would be interesting to see if the conclusion has any probabilistic meaning. See [27], [28] for other very interesting positivity properties connected with orthogonal polynomials that were inspired by probalistic considerations.

3. Other positive coefficient expansions. The previous problem can be thought of as trying to find interesting functions in the cone $\sum_{n=0}^{\infty} \alpha_n p_n(x)$, $\alpha_n \geqq 0$. There is another problem that can be thought of in this way. If $\{g_n(x)\}$ and $\{p_n(x)\}$ are two sequences of orthogonal polynomials we want to know when

$$(3.1) \qquad g_n(x) = \sum_{k=0}^{n} \alpha_{k,n}\, p_k(x), \qquad \alpha_{k,n} \geqq 0.$$

There are many instances of this, both for the classical polynomials and for some non-classical polynomials, but there are no really satisfactory answers to the general question.

For the classical polynomials the following are known.

	$g_n(x)$	$p_n(x)$	
(3.2)	$L_n^{\alpha+\mu}(x)$	$L_n^{\alpha}(x)$	$\mu > 0, \quad \alpha > -1,$
(3.3)	$P_n^{(\alpha+\mu,\alpha+\mu)}(x)$	$P_n^{(\alpha,\alpha)}(x)$	$\mu > 0, \quad \alpha > -1,$
(3.4)	$P_n^{(\alpha+\mu,\beta)}(x)$	$P_n^{(\alpha,\beta)}(x)$	$\mu > 0, \ \alpha,\beta > -1,$
(3.5)	$P_n^{(\alpha,\beta-1)}(x)$	$P_n^{(\alpha,\beta)}(x)$	$\alpha > -1, \quad \beta > 0,$
(3.6)	$P_n^{(\alpha,\beta)}(x)$	$T_n(x) = P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$	
			$\alpha \geqq \beta \geqq -\frac{1}{2}; \ \alpha \geqq \beta+1, -\frac{1}{2} > \beta > -1.$

For $\mu = 1, 2, \ldots$, (3.2), (3.3), (3.4) are special cases of a theorem for general polynomials [2]. The following conjecture was given in [2] and we are still no closer to having a proof or a counter example.

Conjecture. Let $w(x)$ be a measure on $[a,b]$, a finite, $\{p_n(x)\}$ polynomials orthogonal with respect to $w(x)$ on $[a,b]$ normalized by $p_n(a) > 0$. Let $p_n^{\mu}(x)$ be the polynomials orthogonal with respect to $(x-a)^{\mu}\, w(x)$ on $[a,b]$ normalized by $p_n^{\mu}(a) > 0$. Then

$$p_n^{\mu}(x) = \sum_{k=0}^{n} \alpha_{k,n}\, p_k(x), \quad \alpha_{k,n} \geqq 0 .$$

For $\mu = 1, 2, \ldots$, this follows from two results of Christoffel [40]. To add one more piece of evidence to the plausibility of the conjecture we remark that for $\mu = 1$, the $\alpha_{k,n}$ are not only positive but are totally positive. This follows since they are just $c_n\, d_k\, e_{kn}$ with $e_{kn} = 1$, $k \leqq n$, $e_{kn} = 0$, $k > n$, and such sequences are totally positive [26]. As Karlin has remarked [25] often kernels connected with orthogonal polynomials are sign regular of order at most two, but for the classical polynomials they can be sign regular of a much higher order. Hopefully a similar phenomenon is at work here. For Laguerre polynomials the coefficients $\alpha_{k,n}$ are sign regular of some positive order and for $\mu = 1$ and general orthogonal polynomials we have total positivity. Hopefully for general $\mu > 0$ and general orthogonal polynomials we still have positivity. It may be possible to solve the problem when $\mu = \frac{1}{2}$, but the general case seems to be very hard.

Outside of actually computing $\alpha_{k,n}$ the only other published result that I know that gives the nonnegativity of $\alpha_{k,n}$ is [3], where we observed that Jacobi polynomial expansions with non-negative coefficients are closely tied up with the question of when one projective space can be isometrically imbedded in a different projective space. For example it was shown that

$$(3.7) \qquad P_n^{(2\alpha+1,0)}(x) = \sum_{k=0}^{n} \alpha_{k,n} P_k^{(\alpha,-\frac{1}{2})}(x), \quad \alpha_{k,n} \geqq 0,$$

$\alpha = 0, \frac{1}{2}, 1, \ldots$. For other examples see [3]. There was a very plausible conjecture set forth in [3] about when

$$(3.8) \qquad P_n^{(\gamma,\delta)}(x) = \sum_{k=0}^{n} \alpha_{k,n} P_j^{(\alpha,\beta)}, \quad \alpha_{k,n} \geqq 0.$$

The conjecture was $\delta \geqq \beta$ and $(\gamma+1)/(\delta+1) \geqq (\alpha+1)/(\beta+1)$. For $\gamma + 1 < (\alpha+1)(\delta+1)/(\beta+1)$ it was shown that $\alpha_{0,1} < 0$. Unfortunately the conjecture is false and needs to be modified in the following way. We must also assume $\gamma \geqq \delta + 2\alpha - 2\beta$. This follows from an asymptotic formula for ${}_3F_2(-n, n+a, b; c,d; 1)$ of Fields [21] and the expression for $\alpha_{k,n}$ found by Feldheim [19], [33]. We omit the calculation since it adds nothing to our understanding of the problem.

Even if this new conjecture is true we are still a long way from understanding the complete problem (3.8). This is because of the isolated results (3.5). If $(\gamma,\delta) = (\alpha, \beta-\mu)$ and μ is an integer we have $\alpha_{k,n} \geqq 0$, but if μ is not an integer it is easy to check that some $\alpha_{k,n} < 0$. Thus if the conjecture is true then there are also some positive results for $\delta < \beta$. I don't begin to understand the complete answer for $\delta < \beta$, and the best conjecture I can put forth is to add the points (γ,δ) with $\delta < \beta$ and (γ,δ) to the right of the line through (α,β) with slope minus the slope of the line through (α,β) and $(-1,-1)$. But I don't have enough faith in this to conjecture it. What ever the answer turns out to be it will have interesting implicatations for isometric imbeddings of projective spaces. Details will be supplied when some results have been obtained.

There are a number of unpublished results which I would like to mention and refer the reader to the forthcoming papers for the details.

For Jacobi polynomials G. Gasper has shown (3.8) holds if $\beta \geqq \alpha$ and (γ,δ) lies in the region $\gamma \geqq \alpha$, $\beta + \alpha - \gamma \leqq \delta \leqq \beta + \gamma - \alpha$. If the above conjectures are true then the correct region is larger that this.

M.W. Wilson has proven the following theorem [44].

Theorem 4 . Let $g_n(x)$ be orthogonal on E with respect to $d\beta(x)$ and $p_n(x)$ with respect to $d\alpha(x)$. If

$$\int_E p_n(x)\, p_m(x)\, d\beta(x) \leqq 0, \quad n \neq m,$$

then

$$g_n(x) = \sum_{k=0}^{n} \alpha_{k,n}\, p_k(x) \ , \quad \alpha_{k,n} \geqq 0.$$

It is surprising that a result of this type can be used for anything, but Wilson has used it very effectively to investigate a new set of discrete polynomials which approximate to Legendre polynomials both quantitatively and qualitatively better than the classical discrete Tchebycheff polynomials [45].

These new polynomials seem to be very interesting objects and are worth a good deal more investigation. The same is true of the Pollaczek polynomials. Much of the time spent on obtaining new but useless formules for many special functions could be better spent exploring some of the interesting special functions that have not been throughly investigated, primarily because they do not lend themselves to simple explicit formulas. As an example, the dual convolution structure that was discussed in section 2 needs to be proven for all of the Pollaczek polynomials for which it is true, as well as for the associated polynomials which we will discuss next.

We close this section with a theorem which comes from the recurrence formulas.

Theorem 5. Let $p_n(x)$ and $g_n(x)$ satisfy

$$x\, g_n(x) = g_{n+1}(x) + \gamma_n\, g_n(x) + \delta_n\, g_{n-1}(x),\ \delta_0 = 0,\ \delta_n > 0,\ n = 1,2,\ \ldots,$$

$$x\, p_n(x) = p_{n+1}(x) + \alpha_n\, p_n(x) + \beta_n\, p_{n-1}(x),\ \beta_0 = 0,\ \beta_n > 0,\ n = 1,2,\ \ldots\ .$$

Then if

$$(3.9) \qquad \gamma_k \leqq \alpha_n,\quad \delta_{k+1} \leqq \beta_{n+1}, \qquad k = 0,\ 1,\ \ldots,\ n,$$

we have

$$g_n(x) = \sum_{k=0}^{n} \alpha_{k,n}\, p_k(x),\ \alpha_{k,n} \geqq 0$$

See [5] for the proof and a few applications. There are a few classical results contained in this theorem (but not as many as I would like) but there are a few results that follow from it that I can not prove in any other way. We give one typical application.

Consider the associated polynomials to the Legendre polynomials. They are usually written $P_n(\nu,x)$. They satisfy

$$(n + \nu + 1)\ P_{n+1}(\nu,x) - (2n + 2\nu + 1)x\ P_n(\nu,x) + (n+\nu)P_{n-1}(\nu,x) = 0,$$

$P_{-1}(\nu,x) = 0$, $P_0(\nu,x) = 1$. Normalizing as above we have

$$x\ R_n(\nu,x) = R_{n+1}(\nu,x) + \frac{(n+\nu)^2}{(2n+2\nu)^2-1}\ R_{n+1}(\nu,x).$$

By the theorem of Favard and Shohat these polynomials are orthogonal for $\nu > -\frac{1}{2}$. When $\nu \to \infty$ we have

$$\lim_{\nu\to\infty} R_n(\nu,x) = c_n\, U_n(x) = k_n\, P_n^{(\frac{1}{2},\frac{1}{2})}(x)$$

for some c_n, k_n. The functions $P_n(1,x)$ are classical and the formula

$$P_n(1,x) = \sum_{k=0}^{n} \alpha_{k,n}\, P_k(x), \quad \alpha_{k,n} \text{ some simple function,}$$

is due to Christoffel. See Hobson [24]. Recently Barrucand and Dickinson [10] calculated the coefficients in

$$(3.10) \qquad P_n(\nu,x) = \sum_{k=0}^{n} \alpha_{k,n}\, P_k(x).$$

We spare the reader the explicit representation for $\alpha_{k,n}$ since it involves the product of 14 gamma functions which are functions of k and n. In the cases $\nu = 1$ and $\nu = \infty$ it was known that $\alpha_{k,n} \geqq 0$ and a short calculation from their formula shows this is true for $\nu > 0$. This raises the natural question of the coefficients in

$$P_n(\nu,x) = \sum_{k=0}^{n} \alpha_{k,n}\, P_k(\mu,x).$$

Theorem 5 gives us $\alpha_{k,n} \geqq 0$ for $\nu > \mu > -\frac{1}{2}$. In addition to other results of this type given in [5] there are some conjectures given there that do not follow from Theorem 5.

One possible generalization of Theorem 5 was suggested to me in a recent letter from Barrucand. In the special case $\alpha_n = \gamma_n = 0$ it should be sufficient to assume $\beta_n \geqq \delta_n$. Unfortunately this conjecture is false. However the conditions (3.9) are clearly too strong and it should be possible to replace them by some weaker condition, possibly on $B_n = \beta_1 + \ldots + \beta_n$, $D_n = \delta_1 + \ldots + \delta_n$. Again this theorem proves too much, since not only is $\alpha_{k,n} \geqq 0$ proven, but again a type of monotonicity is proven. What is needed is a new method that will prove that numbers are positive without proving that they start out positive and stay positive since they increase.

In addition to the applications to projective spaces given in [3], theorems of this type can be applied in other fields as well. Rivlin and Wilson give an application to numerical analysis [36]. They investigate to what extent the numerical analyst's faith that Tchebycheff polynomials give the most rapidly convergent series is a fact and to what extent it is a myth. It is not true in general (even if it is a useful myth) but they use theorems of the above sort to show that it is true in a large class of Jacobi polynomials for a fairly wide class of functions.

This question can also be asked for the classical discrete orthogonal polynomials. Charlier, Meixner, Hahn, and also for the q polynomials of Hahn. For the Charlier polynomials the formula is well known and some formulas for the Meixner polynomials have been obtained by P.A. Lee in his so far unpublished Ph.D. disseration at Monash.
His results imply that

$$m_n(x;\ \beta,\ c) = \sum_{k=0}^{n} \alpha_{k,n}\, m_k(x;\ \alpha,b) \quad , \ \alpha_{k,n} \geqq 0$$

if $\beta \geqq \alpha > 0$ and $0 < b \leqq c < 1$. See [17] for the definition of $m_n(x;\ \beta,c)$.

I know of no results for the q polynomials but it probably isn't too hard to obtain some. In a letter that I have just received Gasper gives a new recurrence relation for a ${}_3F_2$ and uses this to prove the conjecture given above for Jacobi polynomials for $\gamma + \delta \geqq 0$, $\alpha \geqq \beta$. The most interesting problem of this sort now seems to be to find a better theorem than Theorem 5.

4. Integral representations. Any harmonic analyst learns to look for the dual of any problem he considers. Both of the above problems can be dualized for the classical polynomials and are both quite old. There are probably not extensions to general polynomials, but there are generalizations to solutions of Sturm-Liouville equations. We will not go into this very interesting aspect of the problem .

The second problem dualizes to

$$g_n(x) = \int p_n(x)\, d\mu_x(y) \tag{4.1}$$

with some normalization of $p_n(x)$ and $g_n(x)$ and $d\mu_x(y)$ a non-negative measure on the spectrum of $p_n(x)$. In particular Dirichlet proved that

$$\frac{P_n^{(0,0)}(x)}{P_n^{(0,0)}(1)} = \int_{-1}^{1} \frac{P_n^{(-\frac{1}{2},-\frac{1}{2})}(y)}{P_n^{(-\frac{1}{2},-\frac{1}{2})}(1)}\, d\mu_x(y) , \quad d\mu_x(y) \geqq 0, \tag{4.2}$$

and Mehler changed it to give

$$\frac{P_n^{(0,0)}(x)}{P_n^{(0,0)}(1)} = \int_{-1}^{1} \frac{P_n^{(-\frac{1}{2},\frac{1}{2})}(y)}{P_n^{(-\frac{1}{2},\frac{1}{2})}(1)}\, d\mu_x(y) , \quad d\mu_x(y) \geqq 0. \tag{4.3}$$

Mehler's formula is usually given as

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi} \int_0^{\theta} \frac{\cos(n+\frac{1}{2})\phi\, d\phi}{(\cos\phi-\cos\theta)^{\frac{1}{2}}} \tag{4.4}$$

and Dirichlet's result is seen to be of the form (4.2) only in retrospect. Bateman seems to have been the first to find a simple general theorem.

$$(1-x)^{\alpha+\nu} \frac{P_n^{(\alpha+\nu,\beta-\nu)}(x)}{P_n^{(\alpha+\nu,\beta-\nu)}(1)} = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\Gamma(\nu)} \int_x^1 (1-y)^{\alpha} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} (y-x)^{\nu-1} dy, \tag{4.5}$$

$$\nu > 0,$$

which is

$$\frac{P_n^{(\alpha+\nu,\beta-\nu)}(x)}{P_n^{(\alpha+\nu,\beta-\nu)}(1)} = \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}\, d\mu_x(y), \quad d\mu_x(y) \geqq 0,\ \nu > 0,\ \beta - \nu > -1 \tag{4.6}$$

See [7] for this, as well as a number of other integrals, some of which we will mention again. Also some applications are given there. Somewhat earlier Gegenbauer found an integral for $P_n^{(\alpha,\alpha)}(x)$ which can be used to prove that

$$(4.7)\qquad \frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(a)} = \int_{-1}^{1} \frac{P_n^{(-\frac{1}{2},-\frac{1}{2})}(y)}{P_n^{(-\frac{1}{2},-\frac{1}{2})}(1)}\, d\mu_x(y)\ ,\quad d\mu_x(y) \geqq 0, \alpha > -\tfrac{1}{2}\ .$$

See Feldheim [20]. Feldheim was the first to realize the importance of Gegenbauer's integral and he found an interesting generalization of it. Unfortunately he was killed in the war before he could publish it, or before he found a better formula which does not suffer from the defects of his formula. Feldheim's formula was finally published [20], but in the mean while Vilenkin rediscovered it [41]. This formula is very useful since it can be used to prove

$$(4.8)\qquad \frac{P_n^{(\alpha+\nu,\alpha+\nu)}(x)}{P_n^{(\alpha+\nu,\alpha+\nu)}(1)} = \int_{-1}^{1} \frac{P_n^{(\alpha,\alpha)}(y)}{P_n^{(\alpha,\alpha)}(1)}\, d\mu_x(y),\quad d\mu_x(y),\qquad \nu > 0.$$

Feldheim gives two equivalent forms of his formula. One is far too complicated and the other has a singularity in the measure when $y = 0$ (or $\phi = {}^{\pi}/2$) in his notation). Now there is nothing special about $y = 0$ for $P_n^{(\alpha,\alpha)}(y)$ [the distinguished points are $y = \pm 1$] so there must be another formula that implies the Feldheim-Vilenkin formula which does not have these defects. This formula is given in [7] and is

$$(4.9)\qquad \frac{(1-x)^{\alpha+\mu}}{(\frac{1+x}{2})^{n+\alpha+1}}\ \frac{P_n^{(\alpha+\mu,\beta)}(x)}{P_n^{(\alpha+\mu,\beta)}(1)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 \frac{P_n^{(\alpha,\beta)}(y)\ (y-x)^{\mu-1}(1-y)^{\alpha}dy}{P_n^{(\alpha,\beta)}(1)\ (\frac{1+y}{2})^{n+\alpha+\mu+1}}\ .$$

(4.9) implies

$$(4.10)\qquad \frac{P_n^{(\alpha+\nu,\beta)}(x)}{P_n^{(\alpha+\nu,\beta)}(1)} = \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}\, d\mu_x(y)\ ,\quad d\mu_x(y) \geqq 0,\ \nu > 0.$$

This was not pointed out in [7] because I was unable to prove it then.

Since then Gasper has called my attention to a paper of Bailey [11] in which he calculates the Poisson kernel for Jacobi series. Here is an instance where an explicit formula is very useful, since the positivity of the Poisson kernel is obvious from his formula. Using this and an argument that is the same as in Corollary 1 in [7] we get (4.10) from (4.9) .

Combining (4.10) and (4.6) gives

$$(4.11)\quad \frac{P_n^{(\alpha+\nu,\beta-\mu)}(x)}{P_n^{(\alpha+\nu,\beta-\mu)}(1)} = \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}} \, d\mu_x(y), \quad d\mu_x(y) \geqq 0, \; 0 \leqq \mu \leqq \nu, \; \beta-\mu > -1$$

Since there is the result (4.8) going up the diagonal $\alpha = \beta$ there is probably a result with $\mu < 0$ in (4.11). By analogy with the dual case I would conjecture it is the same region that I conjectured in [3]. Integrals seem to be more regular than series so this conjecture probably does not need the modification we give in section 3. Unfortunately the measure $d\mu_x(y)$ now has a series representation rather than an integral representation and it seems rather hard to settle any of these cases. The only case I have worked out is

$\beta = \alpha + 1$, $\delta = \gamma + 1$. This is proven using (4.8) and

$$(4.11)\quad \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)} - \frac{P_{n+1}^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}} = \frac{(2n+\alpha+\beta+2)\;(1+x)}{2(\beta+1)} \frac{P_n^{(\alpha,\beta+1)}(x)}{P_n^{(\alpha,\,\beta+1)}(-1)} .$$

The result is

$$(4.12)\quad \frac{P_n^{(\gamma,\gamma+1)}(x)}{P_n^{(\gamma,\gamma+1)}(1)} = \int_{-1}^{1} \frac{P_n^{(\alpha,\alpha+1)}(y)}{P_n^{(\alpha,\alpha+1)}(1)} \, d\mu_x(y), \quad d\mu_x(y) \geqq 0, \quad \gamma > \alpha.$$

Using the same type of formula to go in the other direction,

$$(4.13)\quad \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} - \frac{P_{n+1}^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(1)} = \frac{(2n+\alpha+\beta+2)\;(1-x)}{2(\alpha+1)} \; \frac{P_n^{(\alpha+1,\beta)}(x)}{P_n^{(\alpha+1,\beta)}(1)}$$

gives some evidence that

$$(4.14)\quad \frac{P_n^{(\gamma+1,\gamma)}(x)}{P_N^{(\gamma+1,\gamma)}(1)} = \int_{-1}^{1} \frac{P_n^{(\alpha+1,\alpha)}(y)}{P_n^{(\alpha+1,\alpha)}(1)}\, d\mu_x(y)\ ,\ d\mu_x(y),\quad \gamma > \alpha$$

fails, but I have not verified this yet. The easiest case to do should be $\alpha = -\frac{1}{2}$, $\gamma = \frac{1}{2}$.

The result for Laguerre polynomials is well known

$$(4.15)\quad x^{\alpha+\mu}\,\frac{L_n^{\alpha+\mu}(x)}{L_n^{\alpha+\mu}(0)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_0^x (x-y)^{\mu-1}\ y^{\alpha}\ \frac{L_n^{\alpha}(y)\,dy}{L_n^{\alpha}(0)}\ ,\ \mu > 0.$$

and this is clearly

$$(4.16)\quad \frac{L_n^{\alpha+\mu}(x)}{L_n^{\alpha+\mu}(0)} = \int_0^{\infty} \frac{L_n^{\alpha}(y)}{L_n^{\alpha}(0)}\ d\mu_x(y),\quad d\mu_x(y) \geqq 0,\quad \mu > 0.$$

5. Convolution structures. The analogue of the first problem is even more interesting, being an integrated form of the addition formula in the ultraspherical case, an open problem which has interesting consequences in the Jacobi case, and containing a recent theorem of Sarmanov and Bratoeva in the Hermite case.

This last result is probably the most striking so we start with a statement of it. An old classical result of Mehler is the positivity of the Poisson kernel for Hermite series

$$\sum_{n=0}^{\infty} \frac{r^n H_n(x) H_n(y)}{2^n n!} \geqq 0, \qquad -\infty < x, y < \infty, \ -1 < r < 1.$$

O.V. Sarmanov asked the question of when

$$(5.1) \qquad \sum_{n=0}^{\infty} \frac{C_n H_n(x) H_n(y)}{2^n n!} \geqq 0, \qquad -\infty < x, y < \infty.$$

Clearly if

$$(5.2) \qquad C_n = \int_{-1}^{1} t^n d\mu(t), \qquad d\mu(t) \geqq 0,$$

then (5.1) holds. He showed that conversely if (5.1) holds for $-\infty < x, y < \infty$ and $\sum_{n=0}^{\infty} C_n^2 < \infty$ then

$$C_n = \int_{-1}^{1} t^n \, d\mu(t).$$

We will show how this is a natural result to suspect by recalling the same result for some other orthogonal expansions. Then we give a new proof the corresponding result for Laguerre series and show how the Jacobi problem can be attacked. And we close with a weak type of convolution structure for some Laguerre series.

The easiest result is for cosine series. Let

$$(5.3) \qquad f(\theta,\phi) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos n\theta \cos n\phi$$

where $\sum_{n=0}^{\infty} c_n^2 < \infty$. Assume $f(\theta,\phi) \geqq 0$ for $0 \leqq \phi,\theta \leqq \pi$. Then

$$f(\theta) = f(\theta,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos n\,\theta \geqq 0 \text{ and so}$$

$$(5.4) \qquad c_n = \frac{1}{\pi} \int_0^{\pi} f(\theta) \cos n\,\theta \, d\,\theta$$

where $f(\theta) \geqq 0$. Conversely if $f(\theta) \geqq 0$, $f \in L^2(0,\pi)$ and c_n is defined by (5.4) then

$$f(\theta,\phi) = \frac{f(\theta+\phi) + f(\theta-\phi)}{2} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos n\theta \cos n\,\phi \geqq 0.$$

There is a difference in the conditions (5.2) and (5.4) and it is a natural question to see if a result can be found which connects them. We give a result for ultraspherical series which formally reduces to the cosine result when $\lambda \to 0$ and to the Hermite series result when $\lambda \to \infty$.

For ultraspherical series, and in particular for cosine series, we can remove the L^2 condition if we work with distributions. Since it takes very little more work and may be of some interest, we give this slightly more general theorem.

Let $C_n^{\lambda}(x)$ be defined by

$$(1 - 2xr+r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)\, r^n, \qquad \lambda > 0.$$

These polynomials satisfy

$$\int_{-1}^{1} C_n^{\lambda}(x)\, C_m^{\lambda}(x)\, (1-x^2)^{\lambda-\frac{1}{2}}\, dx = \frac{2^{1-2\lambda}\, \pi\, \Gamma(n+2\lambda)}{(\Gamma(\lambda))^2\, (n+\lambda)\, \Gamma(n+1)} \, \delta_{nm}$$

and

$$C_n^{\lambda}(1) = \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)\, \Gamma(n+1)} .$$

If $\int_{-1}^{1} |f(x)|\,(1-x^2)^{\lambda-\frac{1}{2}}\, dx < \infty$ we define

$$a_n = \int_{-1}^{1} f(x)\, \frac{C_n^{\lambda}(x)}{C_n^{\lambda}(1)}\, d\,\nu_{\lambda}(x)$$

where

$$\nu_\lambda(x) = \frac{1}{c_\lambda} \int_{-1}^{x} (1-t^2)^{\lambda-\frac{1}{2}} \, dt,$$

$$c_\lambda = \int_{-1}^{1} (1-t^2)^{\lambda-\frac{1}{2}} \, dt.$$

Then

$$f(x) \sim \sum_{n=0}^{\infty} a_n \frac{(n+\lambda)}{\lambda} C_n^\lambda(x).$$

C_n^λ satisfies the differential equation

$$\frac{d}{dx}\left[(1-x^2)^{\lambda+\frac{1}{2}} \frac{d}{dx} C_n^\lambda(x)\right] + n(n+2\lambda)\, C_n^\lambda(x)(1-x^2)^{\lambda-\frac{1}{2}} = 0.$$

If $f(x)$ is infinitely differentiable and $f^{(k)}(\pm 1) = 0$ for all k then a standard argument (integrating by parts) gives

$$a_n = O(n^{-k}), \qquad k = 1, 2, \ldots .$$

Parseval's theorem for ultraspherical series is

$$\int_{-1}^{1} f(x)\, g(x)\, d\nu_\lambda(x) = \sum_{n=0}^{\infty} \frac{a_n b_n (n+\lambda)\, \Gamma(n+2\lambda)\pi}{2^{2\lambda-1}\left[\Gamma(\lambda+1)\right]^2 \Gamma(n+1)}$$

where

$$b_n = \int_{-1}^{1} g(x) \frac{C_n^\lambda(x)}{C_n^\lambda(1)} \, d\nu_\lambda(x) .$$

Following Bochner [13] we define

$$f(x;y) = \frac{1}{c_{\lambda-\frac{1}{2}}} \int_{-1}^{1} f(x\,y + \sqrt{1-x^2}\,\sqrt{1-y^2}\,z)(1-z^2)^{\lambda-1} \, dz.$$

It is clear that $f(x;y) \geq 0$ if $f(z) \geq 0$, $-1 \leq z \leq 1$. Also from the addition theorem for ultraspherical polynomials

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} \frac{C_n^\lambda(y)}{C_n^\lambda(1)} = \frac{C_n^\lambda(x;y)}{C_n^\lambda(1)} = \frac{1}{c_{\lambda-\frac{1}{2}}} \int_{-1}^{1} \frac{C_n^\lambda(xy+\sqrt{1-x^2}\,\sqrt{1-y^2}\,z)}{C_n^\lambda(1)} (1-z^2)^{\lambda-1} dz.$$

Thus

$$f(x;y) \sim \sum_{n=0}^{\infty} a_n \left(\frac{n+\lambda}{\lambda}\right) C_n^{\lambda}(x)\ C_n^{\lambda}(y)/C_n^{\lambda}(1).$$

We define a distribtuion T as a linear functional on the space of C^{∞} functions $f(x)$ with $f^{(k)}(\pm 1) = 0$, $k = 0, 1, \ldots$ by Parseval's theorem. If b_n is a sequence of real numbers satisfying $|b_n| \leq A\, n^k$ for some A and k then define

$$Tf = \sum_{n=0}^{\infty} \frac{a_n\, b_n (n+\lambda)\, \Gamma(n+2\lambda)\, \pi}{2^{2\lambda-1} \left[\Gamma(\lambda+1)\right]^2\, \Gamma(n+1)} \quad .$$

This series converges since $a_n = O(n^{-j})$ for every j.

We call a distribution non-negative if $Tf \geq 0$ for all $f \in \mathcal{S}$, $f(x) \geq 0$, $-1 \leq x \leq 1$. We define the generalized translation T_y of T as the distribution defined by

$$T_y f = \sum_{n=0}^{\infty} \frac{a_n\, b_n (n+\lambda)\, \Gamma(n+2\lambda)\pi}{2^{2\lambda-1}\left[\Gamma(\lambda+1)\right]^2\, \Gamma(n+1)}\ \frac{C_n^{\lambda}(y)}{C_n^{\lambda}(1)} = Tf(\,.\,;y).$$

Then it is clear from the above that $T_y \geq 0$, $-1 \leq y \leq 1$, if and only if $T_1 = T \geq 0$.

If the distribution T comes from a measure, i.e.

$$b_n = \int_{-1}^{1} \frac{C_n^{\lambda}(x)}{C_n^{\lambda}(1)}\, d\mu(x) \ ,$$

then $T \geq 0$ if and only if $d\mu \geq 0$, i.e. $\int_E d\mu(x) \geq 0$ for all Borel sets E.

Thus if

$$(5.5) \qquad f(x,y) = \sum_{n=0}^{\infty} a_n \left(\frac{n+\lambda}{\lambda}\right) C_n^{\lambda}(x)\ C_n^{\lambda}(y)/C_n^{\lambda}(1) \geq 0, \ -1 < x,\, y \leq 1$$

then a_n are the Fourier-ultraspherical coefficients of a non-negative function. If other assumptions are made on a_n then there is a non-negative function f which gives the a_n by

$$(5.6) \qquad a_n = \int_{-1}^{1} \frac{C_n^\lambda(x)}{C_n^\lambda(1)} f(x)\, d\nu(x).$$

If we let $\lambda \to 0$ in (5.5) and (5.6) we obtain the cosine result. This follows from

$$(5.7) \qquad \lim_{\lambda\to 0} \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} = \cos n\theta$$

and

$$(5.8) \qquad \lim_{\lambda\to 0} \frac{n+\lambda}{\lambda} C_n^\lambda(\cos\theta) = \begin{cases} 2\cos n, & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases}$$

If we let $\lambda \to \infty$ and use

$$(5.9) \qquad \lim_{\lambda\to\infty} \frac{C_n^\lambda(x)}{C_n^\lambda(1)} = x^n$$

and

$$(5.10) \qquad \lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} C_n^\lambda(x\lambda^{-\frac{1}{2}}) = H_n(x)/n!$$

we formally obtain the Sarmanov and Bratoeva result.

Now consider the corresponding problem for Laguerre series. The $L_n^\alpha(x)$ are defined by

$$(5.11) \qquad \sum_{n=0}^{\infty} L_n^\alpha(x)\, r^n = (1-r)^{-\alpha-1} \exp(-xr/(1-r)), \qquad \alpha > -1.$$

They satisfy

$$\int_0^\infty L_n^\alpha(x)\, L_m^\alpha(x)\, x^\alpha e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \delta_{n,m}$$

<u>Theorem 6</u>. Let

$$(5.12) \qquad f(x,y) = \sum_{n=0}^{\infty} \frac{a_n L_n^\alpha(x) L_n^\alpha(y) \Gamma(n+1)}{\Gamma(n+\alpha+1)} \geq 0, \quad 0 \leqq x,y < \infty$$

Also assume

$$(5.13) \qquad \sum_{n=0}^{\infty} a_n^2 < \infty.$$

Then

$$(5.14) \qquad a_n = \int_0^1 t^n \, d\mu(t)$$

where $d\mu$ is a positive measure with no mass at $t = 1$. Conversely if a_n is given by (5.14) then (5.12) holds.

Some remarks are in order about the condition (5.13). We will assume it so that we can work in L^2 for simplicity. It is possible to replace it with some weaker assumptions at the expense of making the proof harder. The extra material that is needed is in [34]. If we define a_n by (5.14) and want to have (5.13) satisfied it is necessary and sufficient that

$$\int_0^1 (1-t)^{-2} \left[\int_t^1 d\mu(s)\right]^2 dt < \infty.$$

For this and related ℓ^p theorems see [6].

The sufficiency follows from the Hille-Hardy formula

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^\alpha(x) \, L_n^\alpha(y) \, z^n = \frac{(x\,y\,z)^{-\alpha/2}}{1-z} I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right) \exp\left[-\frac{z(x+y)}{1-z}\right]$$

See [40, (5.1.15)]. $I_\alpha(z)$ is a Bessel function of imaginary argument which is given by

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\alpha+2k}}{k! \, \Gamma(k+\alpha+1)}, \quad z \geq 0.$$

For the necessity of (5.12) we need

$$(5.15) \qquad \frac{1}{n!} e^{-t} t^{n+\frac{\alpha}{2}} = \int_0^\infty e^{-y} y^{\alpha/2} L_n^\alpha(y) \, J_\alpha(2(yt)^{1/2}) dy.$$

This is the inverse Hankel transform of

$$(5.16) \qquad e^{-x} x^{\alpha/2} L_n^\alpha(x) = \frac{1}{n!} \int_0^\infty e^{-t} \, t^{n+\alpha/2} \, J_\alpha(2(xt)^{1/2}) dt,$$

which is given in [40, (5.4.1)]. Or it can be proven from the special case $n = 0$ of (5.16) by means of the generating function (5.11).

Finally we need

$$\lim_{x\to\infty} \frac{L_n^\alpha(x)}{n} = \frac{(-1)^n}{n!}, \qquad [40, (5.1.8)]. \tag{5.17}$$

Using (5.15) and the power series for $J_\alpha(u)$ we see that

$$e^{-\frac{t}{x}} \sum_{n=0}^{\infty} \frac{a_n\, t^n\, L_n^\alpha(x)}{\Gamma(n+\alpha+1)\, x^n} = \sum_{n=0}^{\infty} \frac{(-1)^n\, t^n}{\Gamma(n+\alpha+1)n!} \int_0^\infty \frac{y^{n+\alpha} f(x,y) e^{-y}}{x^n}\, dy.$$

If we fix t and let $x \to \infty$ on the left hand side we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, a_n\, t^n}{\Gamma(n+\alpha+1)n!}.$$

On the right hand side we have $\int_0^\infty u^n (xu)^\alpha e^{-xu} x f(x,ux)\, du$ which is $\int_0^\infty u^n\, d\,\mu_x(u)$, where $d\mu_x(u)$ is a positive measure. Since the left hand side converges for all n as $x \to \infty$ so does the right hand side and we get convergence to $d\mu(u) \geqq 0$, since $(x\,u)^\alpha e^{-xu}\, x\, f(x,ux) \geqq 0$ for all x and u.

Thus

$$\sum_{n=0}^{\infty} \frac{(-1)\, a_n\, t^n}{\Gamma(n+\alpha+1)n!} = \sum_{n=0}^{\infty} \frac{(-1)^n\, t^n}{\Gamma(n+\alpha+1)n!} \int_0^\infty u^n\, d\mu(u)$$

so

$$a_n = \int_0^\infty u^n\, d\mu(u).$$

But the series $\sum_{n=0}^{\infty} n!\, u^n \frac{L_n^\alpha(x)\, L_n^\alpha(y)}{\Gamma(n+\alpha+1)}$ dosn't converge if $u \geqq 1$ so

a is given by(5.14) where $d\mu(x)$ is a positive measure with no mass at $x = 1$.

A more complicate proof of this result for Laguerre polynomials was given by I.O. Sarmanov [37].

The Sarmanov - Bratoeva theorem and Theorem 6 show that no convolution structure can exist for Hermite and Laguerre series which has the desirable feature of preserving positivity. Conside, for example, the series

$$(5.18) \qquad \sum_{n=0}^{\infty} a_n H_n(x) H_n(y) H_n(z).$$

If (5.18) is ≥ 0 for all x,y,z, then

$$2^n n! a_n H_n(z) = \int_{-1}^{1} t^n d\mu_z(t)$$

where $d\mu_z(t)$ is a positive measure. Then if $n = 2$ the right hand side is positive unless $d\mu_z(t)$ only has mass at $t = 0$ and the left hand side changes sign unless $a_2 = 0$. Thus the series $\sum_{n=0}^{\infty} a_n H_n(x) H_n(y) H_n(z)$ must change sign for some values of x,y,z unless $a_n = 0$ for $n \geq 1$. In [1] Al-Salam and Carlitz show that (5.18) cannot be of a certain natural form for all x,y,z. The lack of positivity is an even more striking property.

For Jacobi polynomials we need to consider

$$K_r(x,y,z) = \sum_{n=0}^{\infty} r^n h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) P_n^{(\alpha,\beta)}(z)/P_n^{(\alpha,\beta)}(1), \ \alpha \geq \beta$$

where

$$\left[h_n^{(\alpha,\beta)}\right]^{-1} = \int_{-1}^{1} \left[P_n^{(\alpha,\beta)}(x)\right]^2 (1-x)^\alpha (1+x)^\beta dx.$$

To have a convolution structure we need the have

$$(5.19) \qquad \int_{-1}^{1} |K_r(x,y,z)|(1-z)^\alpha(1+z)^\beta dz \leq C , \quad 0 \leq r < 1, \quad -1 \leq x, y \leq 1.$$

For $\alpha \geq \beta \geq -\frac{1}{2}$ this was proven in [9]. If we knew that $K_r(x,y,z) \geq 0$ then we would have (5.19) trivially since

$$\int_{-1}^{1} K_r(x,y,z)(1-z)^\alpha(1+z)^\beta dz = 1$$

by orthogonality.

Using the differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ and a maximum theorem for hyperbolic equations due to Weinberger, [43], it is possible to prove that $K_r(x,y,z) \geq 0$ for $-1 \leq z \leq 1$, $x + y \geq 0$, $\alpha \geq \beta$, $\alpha + \beta + 1 \geq 0$. In the case $\alpha = \beta$ this was sufficient to give $K_r(x,y,z) \geq 0$, $-1 \leq x,y,z \leq 1$, but for $\alpha > \beta$ this is not so. A start toward proving the positivity for $x + y < 0$ can be made as follows. As we remarked before, Bailey [11] proved that

$$K_r(x,y,1) = \sum_{n=0}^{\infty} r^n h_n P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) \geq 0, \qquad \alpha,\beta > -1,$$

Using Bateman's integral, [7] ,

$$(1+y)^{\beta+\mu} \frac{P_n^{(\alpha-\mu,\beta+\mu)}(y)}{P_n^{(\alpha-\mu,\beta+\mu)}(-1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^{1} (1+t)^{\beta} \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(-1)} (y-t)^{\mu-1} dt$$

for $\mu = \alpha - \beta$ we see that

$$0 \leq \sum_{n=0}^{\infty} r^n h_n P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(-1) \frac{P_n^{(\beta,\alpha)}(y)}{P_n^{(\beta,\alpha)}(-1)}$$

$$= \sum_{n=0}^{\infty} r^n h_n P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(-y) \frac{P_n^{(\alpha,\beta)}(-1)}{P_n^{(\alpha,\beta)}(1)} = K_r(x,-y,-1)$$

Unfortunately I have been able to use this to handle the other cases of $x + y < 0$. Since we have the positivity of $K_r(x,y,z)$ for many values of x,y,z for $\alpha \geq \beta$, $\alpha + \beta + 1 + \geq 0$, a reasonable conjecture is that there is a convolution structure for $\alpha + \beta + 1 \geq 0$. The assumption $\alpha \geq \beta$ is unnecessary for if $\beta > \alpha$ we define

$$K_r(x,y,z) = \sum_{n=0}^{\infty} r^n h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) P_n^{(\alpha,\beta)}(z)/P_n^{(\alpha,\beta)}(-1)$$

and then everything we can prove for $\alpha \geq \beta$ can be extended to $\alpha < \beta$. And in both this case and the dual case a convolution structure without necessarily a positive kernel probably exists for $\max(\alpha,\beta) \geq -\frac{1}{2}$.

A stochastic application of the ultraspherical result is given by Lamperti [30]. Other applications are given in [9] and there are other applications that can be given once the positivity is proven for Jacobi polynomials.

For Laguerre series a convolution can be defined in the following way. Watson [42] proved that

$$(5.20)\quad \frac{n!\, L_n^\alpha(x)\, L_n^\alpha(y)}{\Gamma(n+\alpha+1)} = \frac{1}{\sqrt{\pi}} \int_0^\pi e^{-\sqrt{xy}\cos\theta}\, \frac{J_{\alpha-\frac{1}{2}}((xy)^{\frac{1}{2}}\sin\theta)}{\left(\frac{(xy)^{\frac{1}{2}}\sin\theta}{2}\right)^{\frac{1}{2}}}$$

$$L_n^\alpha(x+y+2(xy)^{\frac{1}{2}}\cos\theta)\,d\theta, \quad \alpha > -\tfrac{1}{2}.$$

Recall that

$$(5.21)\qquad L_n^\alpha(0) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}$$

and

$$(5.22)\qquad \int_0^\infty [L_n^\alpha(x)]^2\, x^\alpha e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}$$

If $f(x)$ is integrable on each finite interval and does not grow too fast we can define its Fourier-tagnerre coefficient by

$$(5.23)\qquad \hat{f}(n) = \int_0^\infty f(x)\, \frac{L_n^\alpha(x)}{L_n^\alpha(0)}\, x^\alpha e^{-x}\, dx.$$

Then we have formally

$$f(x) \sim \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \hat{f}(n)\, L_n^\alpha(x).$$

Using (5.20) as a model we define the generalized translate of $f(x)$ by

$$(5.24)\quad f(x;y) = \frac{1}{\sqrt{\pi}} \int_0^\pi f(x+y+2\sqrt{xy}\cos\theta)\, e^{-(xy)^{\frac{1}{2}}\cos\theta}\, \frac{J_{\alpha-\frac{1}{2}}((xy)^{\frac{1}{2}}\sin\theta)}{\left(\frac{(xy)^{\frac{1}{2}}}{2}\sin\theta\right)^{\alpha-\frac{1}{2}}}\, \sin^{2\alpha}\theta\, d\theta.$$

Then f(x;y) has the expansion

(5.25) $$f(x,y) \sim \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \hat{f}(n)\, L_n^{\alpha}(x)\, \frac{L_n^{\alpha}(y)}{L_n^{\alpha}(0)}$$

Define the convolution of f(x) and g(x) by

(5.26) $$h(x) = \int_0^{\infty} f(x;y)\, g(y)\, y^{\alpha} e^{-y}\, dy.$$

Then a simple calculation shows that

(5.27) $$\hat{h}(n) = \hat{f}(n)\, \hat{g}(n).$$

Formally this all makes sense, even defining f(x;y) by (5.25), provided that this series makes sense. McCully [31] found a subspace of L^1 for which this all makes sense in the special case $\alpha = 0$ and the following theorem can be proven in the same way his Theorem 5 was proven.

Theorem 7. Let f(x), g(x) be integrable on each finite interval of $x \geq 0$ and satisfy

(5.28) $$f(x),\ g(x) = O(e^{ax}), \quad a < \tfrac{1}{2},\ x \to \infty$$

Then h(x) defined by (5.26) is also integrable on each finite interval, $h(x) = O(e^{ax})$, $a < \frac{1}{2}$, $x \to \infty$ and (5.27) holds.

The condition (5.28) can probably be weakened slightly but it cannot be weakened to $O(e^{x/2})$, since h(x) does not exist if $f(x) = g(x) = e^{x/2}$. This does not show that it is not possible by some other method to define the convolution of these two function, but it indicates that it is unlikely.

There are still a couple of interesting questions raised by Watson's integral (5.20). The first is to find a substitute for $\alpha = -\frac{1}{2}$, which is equivalent to the case of even Hermite polynomials. By analogy with the ultraspherical case there should be a convolution here also (it should correspond to the cosine case). The other question is not immediately evident, but occurs when we consider the case $\alpha = 0$ in more detail.

In this case (5.20) is the integrated form of an addition formula of Bateman [12]. There is a simple proof of this addition formula due to Carlitz [14]. There should be a corresponding addition formula for $L_n^{\alpha}(x)$ which has (5.20) as its integrated form.

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